ON SCALAR AND VECTOR COVARIANTS OF LINEAR ALGEBRAS*

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INTRODUCTION

1. Relation to the literature. This paper concerns itself with two rather different kinds of covariants of the general linear algebra, which might, for convenience, be distinguished by the adjectives scalar and vector. Consider the general linear algebra† E with the units $e_1, \cdots, e_n$ and with the constants of multiplication γ_{ijk} $(i, j, k = 1, \cdots, n)$, where the general number of the algebra is $X = \sum x_i e_i$. In a previous paper‡ we defined a rational integral covariant C of the algebra E as a rational integral function of the γ's and the x's which possesses the invariantive property whenever the units are subjected to a linear transformation. The paper just mentioned uses such covariants to characterize linear associative algebras in two and three units. The present paper proves for such covariants several fundamental theorems analogous to the fundamental theorems in the theory of invariants for algebraic forms. Since these covariants are isobaric and homogeneous, we can apply here Hilbert's proof of the "finiteness" of the number of covariants of algebraic forms.

We must not, however, content ourselves with the study of such covariants, since they are not sufficient to characterize linear algebras, both associative and non-associative, even when there are only two units. Accordingly, we consider rational integral functions of the γ's, the x's and also the units e_i $(i = 1, \cdots, n)$ which possess the invariantive property. Since these functions involve the e's, we call them *vector covariants* in contradistinction to the functions C mentioned above, which we might call *scalar covariants*. At first sight, it might seem as if we were confronted with a rather inconvenient difficulty for the following reason. In a theory of such covariants we would expect to find treated certain differential operators analogous to the familiar annihilators Ω and O; but Scheffers has shown that, in a linear algebra, a

* Part I was presented to the Society December 27, 1917, under a different title; Part II was presented February 23, 1918.

† For the fundamental definitions in the theory of linear algebras, see Section 2. The most familiar linear algebras are number fields (in particular, the field of reals and the field of ordinary complex numbers) and quaternions.

‡ Annals of Mathematics, second series, vol. 16 (1914), pp. 1–6.

derivative is not uniquely determined unless multiplication is commutative, and we are concerned with algebras both commutative and non-commutative. This Gordian knot can, however, be readily cut by a device. We can, accordingly, derive the annihilators in the approved manner, and hence show that every rational integral vector covariant of the linear algebra is a covariant of the general number of the algebra $X = \sum x_i e_i$ and a suitable set of scalar covariants of the algebra. From this fact flow theorems analogous to those proved for scalar covariants and, in particular, the "finiteness" of vector covariants.

2. Definitions. A *linear algebra* E over the field F is a set of hypercomplex numbers of the form $X = \sum_{i=1}^{n} x_i e_i$, where the *coördinates* x_i range independently over F. Here the *units* e_i are such that $e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k$ $(i, j = 1, \cdots, n)$, where (i) the *constants of multiplication* γ_{ijk} are in F, (ii) the sum of two numbers X and Y of the algebra is $X + Y = \sum_{i=1}^{n}(x_i + y_i) e_i$ and (iii) numbers of the algebra combine under addition and multiplication according to the distributive law. Unless explicitly stated, we do not assume the commutative nor the associative law of multiplication.*

If we have a number $X = \sum x_i e_i$ of an algebra whose coördinates are (ordinary) complex numbers, then we can find a number $Y = \sum y_i e_i \neq 0$ and a scalar ω such that $XY = \omega Y$ if and only if the *right-hand characteristic determinant* $\delta(\omega) \equiv |\sum_{i=1}^{n} \gamma_{ijk} x_i - d_{jk}\omega|$ is zero. Here $d_{jj} = 1$, $d_{jk} = 0$ if $k \neq j$. Similarly, there is a number $Y \neq 0$ and a scalar ω such that $YX = \omega Y$ if and only if the *left-hand* characteristic determinant

$$\delta'(\omega) \equiv |\sum_{i=1}^{n} \gamma_{jik} x_i - d_{jk}\omega|$$

vanishes.

For convenience in the study of covariants of a linear algebra, we introduce the notion of *weight*. If we subject the algebra E to the transformation

$$(\lambda)_r \qquad \begin{aligned} e_r' &= \lambda e_r \qquad (\lambda \neq 0) \\ e_i' &= e_i \qquad (i \neq r), \end{aligned}$$

the γ's are subject to the induced transformation where γ_{ijk} is unaltered when $i, j, k \neq r$, γ_{ir} $(i, j \neq r)$ is multiplied by λ^{-1}, γ_{rir} and γ_{irr} $(i \neq r)$ are unaltered, γ_{rrr} is multiplied by λ, γ_{rjk} and γ_{jrk} $(j, k \neq r)$ by λ and γ_{rrk} $(k \neq r)$ by λ^2. Hence we shall say that γ_{ijk} $(i, j, k \neq r)$ is of weight 0, γ_{ijr} $(i, j \neq r)$ of weight -1 in e_r, γ_{rir} and γ_{irr} $(i \neq r)$ of weight 0, γ_{rrr} of weight 1, γ_{rjk} and γ_{jrk} $(j, k \neq r)$ of weight 1 and γ_{rrk} $(k \neq r)$ of weight 2 in e_r. For every unit e_r we thus assign to γ_{ijk} a weight with respect to that unit, which is counted in the following way: the first subscript, i, contributes $+1$ if $i = r$, other-

* For definitions of a linear associative algebra by a set of independent postulates, see Dickson, these Transactions, vol. 4 (1903), pp. 21–26.

wise 0 ; the second, j, contributes $+1$ if $j = r$, otherwise 0 ; and the third, k, contributes -1 if $k = r$, otherwise 0. The sum of these three partial weights is called the weight of γ_{ijk} with respect to e_r.

For a similar reason, we shall say that e_r is of weight 1 in e_r but of weight 0 in any other unit e_i $(i \neq r)$.

If $X = \sum x_i e_i$ is the general number of the algebra E, then if $x'_1, \cdots, x'_n$ are the coördinates of X when it is expressed in terms of the new units—that is, $X = \sum x'_i e'_i$—then $x'_r = x_r/\lambda$ and $x'_i = x_i$ $(i \neq r)$. For this reason one might be tempted to say that x_r is of weight -1 with respect to e_r and that the other x's are of weight 0 with respect to e_r. In the applications, however, it will be better so to define weight that every x has a weight which is positive or zero with respect to any unit e_r. Accordingly, we shall agree that x_r is of weight 0 in e_r and that x_i $(i \neq r)$ is of weight 1 in e_r.

We shall use the term *total weight* of a term for the sum of the weights of that term in all the units $e_1, \cdots, e_n$. Notice that the total weight of any γ, x_i or e_i is 1, and thus the total weight of a product of a number of γ's, x's and e's is the degree of that term.

If P is the product of a number of γ's (or a number of e's) such that it is of weight w_r in a particular unit, e_r, it is multiplied by λ^{w_r} under the transformation $(\lambda)_r$. If, however, P is the product of a number of x's such that it is of degree s in the x's and of weight w_r in e_r, then under the transformation $(\lambda)_r$ it is multiplied by $\lambda^{w_r - s}$.

Any polynomial in the γ's, the x's and the e's such that all terms have the same weight with respect to a particular unit e_r will be said to be *isobaric with respect to* e_r. If the terms have the same total weight in all the units, the polynomial will be said to be *isobaric on the whole*. If P is a polynomial in the γ's, the x's and the e's which is isobaric with respect to e_r, of weight w_r in e_r and which is homogeneous in the x's of degree s, it is multiplied by $\lambda^{w_r - s}$ under the transformation $(\lambda)_r$.

Part I. Scalar Covariants

3. Some fundamental theorems. The invariancy of I under the transformation

$$T: \qquad e'_l = \sum_{m=1}^{n} a_{lm} e_m \qquad (l = 1, \cdots, n),$$

$$A = |a_{lm}| \neq 0,$$

is expressed by the formula

$$I(\gamma'_{ijk}) = \phi(a_{lm}) I(\gamma_{ijk}) \tag{1}$$

where the γ_{ijk} are the constants of multiplication of the original algebra and

the γ'_{ijk} are the constants of multiplication of the transformed algebra. Here $I(\gamma'_{ijk})$ has been written for $I(\gamma'_{111}, \gamma'_{112}, \cdots, \gamma'_{nnn})$, $\phi(a_{lm})$ for $\phi(a_{11}, a_{12}, \cdots, a_{nn})$ and $I(\gamma_{ijk})$ for $I(\gamma_{111}, \gamma_{112}, \cdots, \gamma_{nnn})$. Hence by applying the special transformation $e'_i = \lambda e_i$ ($\lambda \neq 0$; $i = 1, \cdots, n$), it follows that I is homogeneous in the γ's.

Under this transformation T the γ_{ijk} are subject to an induced transformation such that

(2) $$\sum_p \gamma'_{ijp}\, a_{pq} = \sum_{k,\,l} a_{ik}\, a_{jl}\, \gamma_{klq} \qquad (i, j, q = 1, \cdots, n).$$

If we keep i and j fixed in (2) and let q range from 1 to n, we have n non-homogeneous linear equations in the n unknowns γ'_{ijp} which can be solved uniquely, since $A \neq 0$. Hence each γ'_{ijp} is a homogeneous function of degree $n + 1$ in the a's divided by A.

Accordingly, if $I(\gamma_{ijk})$ is a rational integral invariant of total weight w in the units and hence homogeneous of degree w in the γ's, $I(\gamma'_{ijk})$ is a rational integral homogeneous function of degree $(n + 1)w$ in the a_{lm} divided by A^w. But, since $I(\gamma_{ijk})$ is independent of the a's, this implies that $\phi(a_{lm})$ is a rational integral homogeneous function $\psi(a_{lm})$ of degree $(n + 1)w$ in the a's divided by A^w. That is,

$$I(\gamma'_{ijk}) = \frac{\psi(a_{lm})}{A^w} I(\gamma_{ijk})$$

for every transformation of determinant $A \neq 0$. Hence under the inverse transformation T^{-1}

$$I(\gamma_{ijk}) = \frac{\psi\left(\dfrac{A_{ml}}{A}\right)}{(A^{-1})^w} I(\gamma_{ijk}'),$$

where A_{ml} is the cofactor of a_{ml}. Combining these two results, we see that

$$\psi(a_{lm})\psi\left(\frac{A_{ml}}{A}\right) = 1;$$

or, since ψ is homogeneous of degree $(n + 1)w$,

$$\psi(a_{lm})\psi(A_{ml}) = A^{(n+1)w}.$$

But, since A is irreducible, this can be true only if $\psi(a_{lm}) = kA^u$, where k is some constant. Then $\psi(A_{ml}) = kA'^u$, where $A' = |A_{ml}| = A^{n-1}$, and hence $u = w/n$. By applying the identical transformation, it is evident that $k = 1$, and thus

(3) $$\phi(a_{lm}) = A^{w/n}.$$

This last formula can be proved in the above manner familiar in the classical invariant theory, or it can also be proved for associative algebras by the

following device. If $X = \sum x_i e_i$ is a number of the general n-ary linear algebra $E = (e_1, \cdots, e_n)$, then $EX = E = XE$. That is, if we multiply each unit e_i of E by X, we obtain n linearly independent numbers of E which can be taken as a new set of units. This remark still applies if we restrict ourselves to the general associative algebra. Accordingly, replace e_i by Xe_i $(i = 1, \cdots, n)$. Then

$$e_i' e_j' \equiv (Xe_i)(Xe_j) = \sum_m \left(\sum_{k,l} x_k \gamma_{kjl} \gamma_{ilm}\right)(Xe_m) \quad (i, j = 1, \cdots, n).$$

Thus $I(\gamma_{ijk}')$ is of degree $2w$ in the γ's. But A^r is of degree rn and $I(\gamma_{ijk})$ is of degree w in these γ's. Hence $2w = rn + w$, or $r = w/n$.

Also, if w_i is the weight of I in e_i, then $w_1 = w_2 = \cdots = w_n$. This follows from equations (1) and (3) and the fact that I must be unaltered (except possibly for sign) when any two units, say e_i and e_j, are interchanged.

Since w/n is the weight of I in any particular unit, say e_1, w/n is a positive integer. Notice, moreover, that w is equal to the degree of I, since each γ_{ijk} is of total weight 1 in the units. Thus we have

THEOREM I. *For an n-ary linear algebra $E = (e_1, p \cdots, e_n)$ with constants of multiplication γ_{ijk}, let I be a rational integral invariant of degree d. Then I is homogeneous and isobaric of total weight d in all the units, and d/n is a positive integer. If we subject the units of E to the transformation T of determinant $A \neq 0$, carrying E into the algebra E′ with constants of multiplication γ_{ijk}', then*

$$I(\gamma_{ijk}') = A^{d/n} I(\gamma_{ijk}).$$

The covariancy of the rational integral function C under the transformation T is expressed by the formula

$$C(\gamma_{ijk}'; x_l') = \phi(a_{lm}) C(\gamma_{ijk}; x_l).$$

Such a covariant is not necessarily homogeneous in the coördinates x_i of the general number of the algebra, but it is the sum of covariants C_k such that each C_k is homogeneous in the x's. Accordingly, we may, without loss of generality, restrict ourselves to the study of homogeneous covariants. Every covariant C which is homogeneous in the x's is also homogeneous in the γ's. If C is homogeneous of degree s in the x's and of degree d in the γ_{ijk}, then $C(\gamma_{ijk}'; x_l')$ is a rational integral homogeneous function of degree $(n+1)d + (n-1)s$ in the a's, divided by A^{d+s}. Thus $\phi(a_{lm})$ in this case is a rational integral homogeneous function, ψ, of the a's of degree $(n+1)d + (n-1)s$ divided by A^{d+s}, and consequently

$$\psi(A_{ml})\psi(a_{lm}) = A^{(n+1)d+(n-1)s}.$$

Thence it follows that

$$\phi(a_{lm}) = A^{(d-s)/n}.$$

When we subject E to the special transformation

$$e_i' = e_i\,(i \neq n), \qquad e_n' = \lambda e_n\,(\lambda \neq 0),$$

each term of $C(\gamma'_{ijk})$ is equal to the corresponding term of $C(\gamma_{ijk})$ multiplied by $\lambda^{w_n - s}$, where w_n is the weight of that term in e_n. This follows at once from the remarks at the end of section 2, since this is the transformation $(\lambda)_n$. But under this transformation, $C(\gamma_{ijk})$ is multiplied by $\lambda^{(d-s)/n}$ and thus $w_n - s = (d-s)/n$ or $w_n = [d + (n-1)s]/n$. It follows that the index of the power of A—namely $(d-s)/n$—is an integer. Moreover, it is not negative, for it is the weight of the coefficient of a term of C which is independent of x_n. Now such a term actually occurs in C, as otherwise x_n would be a factor of C, and hence (in view of the covariancy of C) every linear function of the x's would be a factor of C, which is impossible. Thus we have proved

THEOREM II. *For an n-ary linear algebra $E = (e_1, \cdots, e_n)$ where the constants of multiplication are γ_{ijk} and the general number of the algebra is $X = \sum x_i e_i$, let C be a rational integral covariant of degree d in the γ_{ijk} and degree s in the x_i. Then C is isobaric of total weight $d + (n-1)s$ in all the units. If C is homogeneous in the x's, it is homogeneous in the γ's. Moreover, if we subject the units of E to the transformation T of determinant $A \neq 0$, carrying E into the algebra E' with constants of multiplication γ'_{ijk}, then*

$$C(\gamma'_{ijk};\, x'_l) = A^{(d-s)/n}\, C(\gamma_{ijk};\, x_l).$$

*The index, $(d-s)/n$, is an integer, positive or zero.**

4. Fundamental sets of invariants and covariants for $n = 1, 2$. When $n = 1$, the multiplication table is $e_1 e_1 = \gamma e_1$ and the general number of the algebra is $x_1 e_1$. Hence $\delta(\omega) = \delta'(\omega) = \omega - \gamma x_1$; and the only rational integral invariants in this case are powers of γ, and the only rational integral scalar covariants of the unary algebra are all polynomials in γ and x_1.

When $n = 2$, the multiplication table is of the form $e_i e_j = \gamma_{ij1} e_1 + \gamma_{ij2} e_2$ $(i, j = 1, 2)$ and the general number of the algebra is $X = x_1 e_1 + x_2 e_2$. Hence

$$\delta(\omega) = \omega^2 - \omega[\Gamma_1 x_1 + \Gamma_2 x_2] + [\Gamma_5 x_1^2 + \Gamma_9 x_1 x_2 + \Gamma_6 x_2^2],$$

$$\delta'(\omega) = \omega^2 - \omega[\Gamma_3 x_1 + \Gamma_4 x_2] + \Gamma_7 x_1^2 + \Gamma_{10} x_1 x_2 + \Gamma_8 x_2^2],$$

where

$$\Gamma_1 = \gamma_{111} + \gamma_{122}, \qquad \Gamma_2 = \gamma_{211} + \gamma_{222},$$

$$\Gamma_3 = \gamma_{111} + \gamma_{212}, \qquad \Gamma_4 = \gamma_{121} + \gamma_{222},$$

* By the use of this theorem, we can readily prove the following corollary: Every rational integral invariant and covariant of the general linear algebra vanishes for a nilpotent algebra. This theorem was stated and proved for invariants in a paper in the Annals of Mathematics, second series, vol. 18 (1916), p. 84, and the proof for covariants proceeds in essentially the same way.

$$\Gamma_5 = \gamma_{111}\gamma_{122} - \gamma_{112}\gamma_{121}, \qquad \Gamma_6 = \gamma_{211}\gamma_{222} - \gamma_{212}\gamma_{221},$$

$$\Gamma_7 = \gamma_{111}\gamma_{212} - \gamma_{112}\gamma_{211}, \qquad \Gamma_8 = \gamma_{121}\gamma_{222} - \gamma_{122}\gamma_{221},$$

$$\Gamma_9 = \gamma_{111}\gamma_{222} + \gamma_{211}\gamma_{122} - \gamma_{121}\gamma_{212} - \gamma_{112}\gamma_{221},$$

$$\Gamma_{10} = \gamma_{111}\gamma_{222} + \gamma_{121}\gamma_{212} - \gamma_{122}\gamma_{211} - \gamma_{112}\gamma_{221}.$$

We may prove without any difficulty that, for a binary algebra, every homogeneous isobaric function of the constants of multiplication for which $w_1 = w_2$ and which is annihilated by

$$U_{12} = (\gamma_{121} + \gamma_{211})\frac{\partial}{\partial\gamma_{111}} + (\gamma_{222} - \gamma_{121})\frac{\partial}{\partial\gamma_{122}} + (\gamma_{222} - \gamma_{211})\frac{\partial}{\partial\gamma_{212}}$$
$$+ [(\gamma_{212} + \gamma_{122}) - \gamma_{111}]\frac{\partial}{\partial\gamma_{112}} + \gamma_{221}\left[\frac{\partial}{\partial\gamma_{121}} + \frac{\partial}{\partial\gamma_{211}} - \frac{\partial}{\partial\gamma_{222}}\right]$$

is an invariant of the algebra. We can prove a similar theorem for scalar covariants. Accordingly, one may readily determine the rational integral scalar covariants of a given weight, and so one might hope to find a fundamental set of covariants for the binary algebra.

One can, however, avoid such troubles in view of

THEOREM III. *Every scalar covariant of the binary algebra $E = (e_1, e_2)$ where the constants of multiplication are γ_{ijk} and where the general number of the algebra is $X = x_1 e_1 + x_2 e_2$ is a covariant of the characteristic determinants δ and δ'.*

We want to show that any rational integral function $C(\gamma_{ijk}\,;\,x_l)$ of the eight γ's and the x's which has the invariantive property under the group of linear transformations on e_1 and e_2 is a function of $\Gamma_1, \cdots, \Gamma_8$ and the x's which is a covariant of the system of forms $\Gamma_1 x_1 + \Gamma_2 x_2$, $\Gamma_3 x_1 + \Gamma_4 x_2$, $\Gamma_5 x_1^2 + \Gamma_9 x_1 x_2 + \Gamma_6 x_2^2$, $\Gamma_7 x_1^2 + \Gamma_{10} x_1 x_2 + \Gamma_8 x_2^2$ under the group of linear transformations on x_1 and x_2. Since x_1 and x_2 are contragredient to e_1 and e_2, $C(\gamma_{ijk}\,;\,x_l)$ has the invariantive property under the group of linear transformations on x_1 and x_2 where the γ's are transformed according to (2). Thus we can prove our theorem if we can show that any function of the γ's is a function of $\Gamma_1, \cdots, \Gamma_8$. Now this follows from the fact that the Jacobian of $\Gamma_1, \cdots, \Gamma_8$ is

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \gamma_{122} & 0 & -\gamma_{121} & 0 & -\gamma_{112} & 0 & 0 & \gamma_{111} \\ 0 & \gamma_{211} & 0 & -\gamma_{212} & 0 & -\gamma_{221} & \gamma_{222} & 0 \\ \gamma_{212} & 0 & -\gamma_{211} & 0 & 0 & \gamma_{111} & -\gamma_{112} & 0 \\ 0 & \gamma_{121} & 0 & -\gamma_{122} & \gamma_{222} & 0 & 0 & -\gamma_{221} \end{vmatrix},$$

which is not identically zero, since the terms free of γ_{112}, γ_{111}, γ_{222} and γ_{221} equal the four-rowed determinant in the lower left-hand corner.

5. Finiteness of the rational, integral scalar covariants. In view of the homogeneity and isobarism of rational integral scalar covariants, we can readily prove that they are all expressible as polynomials in a finite subset. For, by Hilbert's theorem about an infinite sequence of polynomials,* any such covariant C can be expressed in the form

$$C(\gamma;\, x) = \sum K_j(\gamma;\, x)\, C_j(\gamma;\, x) \tag{5}$$

where the C_j are scalar covariants of the algebra which are the same for every C. Since C and C_j are homogeneous and isobaric, we may suppose that the K_j are also. If we subject E to the transformation T of determinant $A \neq 0$, (5) is replaced by

$$A^w\, C(\gamma;\, x) = \sum A^{w_j} K_j(\gamma';\, x')\, C_j(\gamma;\, x), \tag{6}$$

where C is of index w and C_j is of index w_j. Now operate on both sides of (6) with

$$\Omega_a \equiv \sum (\pm\, i_1 \cdots i_n) \frac{\partial^n}{\partial a_{1i_1} \cdots \partial a_{ni_n}}$$

until both sides of the resulting equation no longer involve the a's. Then, by a well-known theorem of Hilbert's (which is essentially one given by Mertens and Gordan†), (6) becomes

$$NC(\gamma;\, x) = \sum N_j\, C_j'(\gamma;\, x)\, C_j(\gamma;\, x),$$

where the C_j' are scalar covariants of the algebra, and N and N_j are constants different from zero. Thus we prove by induction

THEOREM IV. *For the general n-ary linear algebra there is a finite set $C_1, \cdots, C_m$ of scalar covariants such that every rational integral scalar covariant of the algebra is a polynomial in $C_1, \cdots, C_m$.*

COROLLARY. For the general n-ary linear algebra there is a finite set $B_1, \cdots, B_p$ of rational scalar covariants such that every rational integral scalar covariant of the algebra is a covariant of this set of algebraic forms. For convenience, such a *set of covariants* will be called *basic*.

* "*Ueber die Theorie der algebraischen Formen,*" Mathematische Annalen, vol. 36 (1890), p. 474.

† Hilbert, loc. cit., p. 524; Gordan, *Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist,* Journal für Mathematik, vol. 69 (1868), pp. 323–354; *Die simultanen Systeme binärer Formen,* Mathematische Annalen, vol. 2 (1870), pp. 227–280; *Invariantentheorie* (1887), vol. II, § 9; Mertens, *Beweis dass alle Invarianten und Covarianten* . . ., Journal für Mathematik, vol. 100 (1887), pp. 223–230.

PART II. VECTOR COVARIANTS

6. Preliminary remarks. As we pointed out in the introduction, scalar covariants are not sufficient to characterize linear algebras—in fact, they are not sufficient to characterize potent algebras in 2 units, for the two algebras

$$e_1 e_1 = e_1, \qquad e_1 e_2 = e_2 e_1 = 0, \qquad e_2 e_2 = \gamma e_1 \quad (\gamma > 0);$$

$$e_1 e_1 = e_1, \qquad e_1 e_2 = e_2 e_1 = 0, \qquad e_2 e_2 = 0$$

are not equivalent with respect to the field of reals, and yet for both algebras $\delta = \delta' = \omega(\omega - x_1)$ and both have the same rank and the same index.

Accordingly we shall now consider vector covariants. By a rational integral function of the γ's, the x's and the e's we shall understand any linear combination of a finite number of products of units of the algebra where the coefficients are polynomials in the γ's and x's. It must be borne in mind that, since multiplication is not in general commutative nor associative, we cannot, for instance, combine into one term two such as $e_1[(e_1 e_2)e_3]$ and $(e_1 e_1)(e_2 e_3)$. That is, in some terms the multiplication is to be carried out in one order and in other terms in a different order. Every rational integral vector covariant V is, however, the sum of a finite number of vector covariants V_k such that, in any one V_k, the multiplication indicated in every term is to be carried out in the same order.

7. Fundamental properties. We can readily show that if a vector covariant is homogeneous in the units, e_i, and homogeneous in the x_i, it is homogeneous in the γ_{ijk}. Moreover it is isobaric in any unit e_i, and its weight in e_j is equal to its weight in e_i $(i, j = 1, \cdots, n)$. From these properties we can readily prove the analogue of Theorem II, which we state here merely for the sake of completeness.

THEOREM V. *For an n-ary linear algebra* $E = (e_1, \cdots, e_n)$ *where the constants of multiplication are* γ_{ijk} *and the general number of the algebra is* $X = \sum x_i e_i$, *let* V *be a rational integral vector covariant of degree* d *in the* γ_{ijk}, *of degree* s *in the* x's *and of degree* v *in the* e's. *Then* V *is isobaric of weight* $w = d + (n-1)s + v$ *in the units altogether. If* V *is homogeneous in the* x's *and the* e's, *it is homogeneous in the* γ's. *Furthermore, if we subject the units of* E *to the transformation* T *of determinant* $A \neq 0$, *carrying* E *into the algebra* $E' = (e_1', \cdots, e_n')$ *where the general number of the algebra is* $X' = \sum x_i' e_i'$ *and the constants of multiplication are* γ_{ijk}', *then*

$$V(\gamma_{ijk}'; x_l'; e_m') = A^{(d-s+v)/n} V(\gamma_{ijk}; x_l; e_m).$$

Moreover the index $(d - s + v)/n$ *is an integer, positive or zero.*

8. Annihilators for vector covariants. Since we are dealing with a *linear* algebra, every vector covariant whose degree in the units is greater than 2

can be made linear in the units. Henceforth, unless otherwise specified, we shall think of every vector covariant as linear in the units.

First consider absolute vector covariants of degree 1 in the x's. Now such a covariant is of the form

$$\sum_{i=1}^{n} [x_1 \phi_{1i}(\gamma) + x_2 \phi_{2i}(\gamma) + \cdots + x_n \phi_{ni}(\gamma)] e_i \tag{7}$$

where each ϕ_{ii} is a polynomial of weight zero in each unit, and each ϕ_{ij} $(i \neq j)$ is a polynomial of weight 1 in e_i, of weight -1 in e_j, and of weight zero in the other units. But any polynomial in the γ's of degree d is of total weight d in the e's altogether. Hence each $\phi_{ij} = 0$ $(i \neq j)$ and each ϕ_{ii} is a constant, c_i. But, since (7) is unaltered if we interchange any two units, the c_i are all equal, and thus a vector covariant which is of the first degree in the x's is necessarily a constant multiple of the general number of the algebra.

Next, consider absolute vector covariants of degree 2 in the x's ; such a covariant is necessarily of the form

$$\sum_{i=1}^{n} [x_i^2 \phi_{iii}(\gamma) + x_i \sum_{j \neq i} x_j \phi_{iji}(\gamma) + \sum_{j,\, k \neq i} x_j x_k \phi_{jki}(\gamma)] e_i \tag{8}$$

where each ϕ_{iii} is of weight 1 in e_i and of weight zero in e_l $(l \neq i)$, each ϕ_{iji} $(i \neq j)$ is of weight 1 in e_j and of weight zero in e_l $(l \neq j)$, and each ϕ_{jki} $(j, k \neq i)$ is of weight -1 in e_i, of weight 1 in e_j and in e_k and of weight zero in e_l $(l \neq i, j, k)$. Accordingly each ϕ is of degree 1 in the γ's. In particular,

$$\phi_{111} = \alpha\gamma_{111} + \beta(\gamma_{122} + \gamma_{133} + \gamma_{1nn}) + \gamma(\gamma_{212} + \cdots + \gamma_{n1n}),$$

$$\phi_{121} = \delta\gamma_{121} + \epsilon(\gamma_{323} + \cdots + \gamma_{n2n}) + \zeta\gamma_{211} + \eta(\gamma_{233} + \cdots + \gamma_{2nn}) + \theta\gamma_{222},$$

$$\phi_{231} = \kappa\gamma_{231} + \lambda\gamma_{321}, \qquad \phi_{221} = \mu\gamma_{221},$$

and any ϕ_{iii} is obtained from ϕ_{111} by interchanging the subscripts 1 and i; any ϕ_{iji} $(j \neq i)$ is obtained from ϕ_{121} by interchanging 1 and i, and 2 and j; and so forth. Hence the difference between (8) and $\beta C_{n-1} X + \gamma C'_{n-1} X + (\alpha - \beta - \gamma) X^2$ is a covariant of the type (8) where the α, β, γ are zero. Here C_{n-1} and C'_{n-1} are the coefficients of ω^{n-1} in $\delta(\omega)$ and $\delta'(\omega)$ respectively. Hence consider the covariant (8) where α, β, γ are all zero. Since such a covariant must be unaltered under the transformation

$$e_1' = e_1 + e_2, \qquad e_i = e_i \; (i \neq 1),$$

we must have $\delta = \epsilon = \zeta = \eta = \theta = \kappa = \lambda = \mu = 0$. Therefore every ra-

tional integral absolute vector covariant of degree 2 in the x's is a homogeneous polynomial of degree 2 in C_{n-1}, C'_{n-1} and $X = \sum x_i e_i$.

More generally we have

THEOREM VI. *Every rational integral vector covariant of the n-ary linear algebra is a covariant of the general number of the algebra and any basic set of scalar covariants of the algebra.*

We prove this by using a set of annihilators for vector covariants—differential operators analogous to the familiar Aronhold operators Ω and O. In deriving such operators for the scalar covariants, we used Taylor's theorem with a remainder for polynomials in the γ's and the x's, and so we now seem to be confronted with the necessity of inquiring if this theorem (or one closely analogous) holds for polynomials in the γ's, the x's and the e's—which we might, for brevity, call vector polynomials. This brings forward the question of a differential calculus for such *vector* polynomials.

Now Scheffers* has considered the theory of functions of hypercomplex variables. In a linear algebra over the field C of ordinary complex numbers in which division is generally possible, he considers functions of the form $f = \sum f_i e_i$ where each f_i is an analytic function of the x_j $(j = 1, \cdots, n)$. When

$$dx = \sum_i dx_i e_i,$$

$$df = \sum_{i,j} \frac{\partial f_i}{\partial x_k} dx_k e_i.$$

Then Scheffers shows that such a linear algebra which possesses a principal unit permits of functions f with derivatives with respect to x which are independent of $\Delta x_1, \cdots, \Delta x_n$—that is, independent of the "direction" in which Δx approaches zero—if and only if multiplication is commutative.

It would seem, then, as if we could not hope to find annihilators for vector covariants of the *general* n-ary linear algebra. There are, however, several ways of avoiding this little difficulty.

As we pointed out at the beginning of this section, we can think of a vector covariant as linear in the e's; and so, to derive the desired annihilators, we need only to consider such derivatives as

$$\frac{\partial e_1}{\partial e_1}, \frac{\partial e_2}{\partial e_1}, \cdots.$$

With this end in view, let us consider the linear function $\phi(e_1) = e_1$. Re-

* *Verallgemeinerung der Grundlagen der gewöhnlichen complexen Functionen*, Leipziger Berichte, vol. 45 (1893), pp. 828–848; vol. 46 (1894), pp. 120–134. There are abstracts of these articles in Paris Comptes Rendus, vol. 116 (1893), pp. 1114–1117, 1242–1244.

placing e_1 by $e_1 + \Delta e_1$, where the increment Δe_1 is a scalar multiple of e_2, say me_2, we have $\Delta\phi = \Delta e_1$. Now divide on the right by Δe_1, i. e. find a number u such that $u \cdot \Delta e_1 = \Delta e_1$. If we are dealing with any *particular* algebra, this number u depends on *that* algebra; but we are interested in the *general* algebra, and thus we want a number u such that $u \cdot \Delta e_1 = \Delta e_1$ (or $u \cdot e_2 = e_2$) for *every* n-ary linear algebra. One such number u is the scalar 1. If we take $u = 1$, $de_1/de_1 = 1$. With a similar understanding,

$$\frac{\partial e_2}{\partial e_1} = 0,$$

$$\frac{\partial e_3}{\partial e_1} = 0, \qquad \cdots.$$

Similarly for the derivatives with respect to the other units.

The advantage of defining these derivatives in this way is that we can differentiate with respect to any unit, such as e_1, in a purely *formal* manner—precisely as if we were differentiating with respect to a scalar. Thus a vector covariant which is linear in the units is annihilated by the differential operators

$$U_{ij} + \sum \left(\frac{\partial x'_k}{\partial a_{ij}} \right) \cdot \frac{\partial}{\partial x_k} + \sum \left(\frac{\partial e'_k}{\partial a_{ij}} \right) \cdot \frac{\partial}{\partial e_k} \quad (i, j = 1, \cdots, n),$$

where the U_{ij} are the annihilators for the invariants of the n-ary algebra.

Or we may reason as follows.

Every vector covariant when linear in the units has these annihilators, since such a covariant has the invariantive property when the e's are replaced by a set of cogredient scalars.

Theorem VI now follows at once from the fact that the annihilators for the vector covariants of the algebra are the same as those for covariants of a basic set of scalar covariants of the algebra and the general number of the algebra, $\sum x_i e_i$.

9. Finiteness of vector covariants. We now readily prove

THEOREM VII. *For the general n-ary linear algebra there is a finite set V_1, $\cdots$, V_m of rational integral vector covariants such that every vector covariant of the algebra is a linear function of $V_1, \cdots, V_m$.*

For every vector covariant of a linear algebra may be made linear in the e's. Now thinking of the e's as replaced by a set of cogredient scalars, which we shall call the y's, we have a set of rational integral functions of the variables x_i $(i = 1, \cdots, n)$ and the contragredient variables y_i $(i = 1, \cdots, n)$ together with the n^3 parameters γ_{ijk} $(i_{ij}, k = 1, \cdots, n)$ which are homogeneous and isobaric and which have the invariantive property under the total linear

group on the n variables x_i. Accordingly Hilbert's "finiteness" proof applies. Or this theorem can be proved as a corollary of Theorem VI.

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